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Quantum mechanics for a vector particle in the magnetic field on 4-dimensional sphere

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Abstract

Quantum-mechanical wave equation for a particle with spin 1 is investigated in presence of external magnetic field in spaces with non-Euclidean geometry with constant positive curvature. Separation of the variable is performed; differential equations in the variable r are solved in hypergeometric functions. The study of z-dependence of the wave function has been reduced to a system of three linked ordinary differential 2-nd order equations; till now the system in z variable is not solved.

1. Introduction, setting the problem

In the present paper, we consider a quantum-mechanical problem a particle with spin 1 described by the Duffin–Kemmer in 3-dimensional Riemann space model in presence of the external magnetic field – relevant publications see in [1–30].

Initial matrix wave equation of Duffin–Kemmer for a spin 1 particle has the for (we adhere notation [31])

$$\left\{ \beta^c [\; i\hbar \; (\; e^{\beta}_{(c)} \partial_\beta + \frac{1}{2} J^{ab} \gamma_{abc} \;) \; - \; \frac{e}{c} A_c \;] \; - \; mc \right\} \Psi = 0 \; , \qquad (1.1)$$

where γ_{abc} stand for Ricci rotation coefficients

$$\gamma_{bac} = -\gamma_{abc} = -e_{(b)\beta;\alpha}\, e^{\beta}_{(a)} e^{\alpha}_{(c)},$$

$A_a = e^{\beta}_{(a)} A_\beta$ are tetrad components of an electromagnetic 4-vector A_β; $J^{ab} = (\beta^a \beta^b - \beta^b \beta^a)$ stand for generators of 10-dimensional representation of the Lorentz group. Below we will use shortened notation $e/c\hbar \Longrightarrow e,\ mc/\hbar \Longrightarrow M$.

In Olevsky paper [32] under the number XI the following coordinates are were specified

$$dS^2 = c^2 dt^2 - \rho^2 \, [\, \cos^2 z (dr^2 + \sin^2 r \, d\phi^2) + dz^2 \,] \,,$$
$$z \in [-\pi/2, +\pi/2] \,, \qquad r \in [0, +\pi], \qquad \phi \in [0, 2\pi] \,. \tag{1.2}$$

Generalization of the concept of an uniform magnetic field for the curved model S_3 is given by the following potential

$$A_\phi = -2B \sin^2 \frac{r}{2} = B \, (\cos r - 1) \,. \tag{1.3}$$

To this potential there correspond a single non-vanishing component of the electromagnetic tensor $F_{\phi r} = \partial_\phi A_r - \partial_r A_\phi = B \sin r$; this tensor satisfies Maxwell equations in S_3.

Let us consider eq. (1.3) in the space S_3. To cylindric coordinates $x^\alpha = (t, r, \phi, z)$ there corresponds the tetrad

$$e^{\beta}_{(a)}(x) = \left| \begin{array}{llll} 1 & 0 & 0 & 0 \\ 0 & \cos^{-1} z & 0 & 0 \\ 0 & 0 & \cos^{-1} z \ \sin^{-1} r & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| . \tag{1.4}$$

Relevant Christoffel symbols and Ricci rotation coefficients are

$$\Gamma^{r}{}_{jk} = \left| \begin{array}{ccc} 0 & 0 & -\mathrm{tg}\ z \\ 0 & -\sin r \cos r & 0 \\ -\tan z & 0 & 0 \end{array} \right| ,$$

$$\Gamma^{\phi}{}_{jk} = \left| \begin{array}{ccc} 0 & \cot r & 0 \\ \cot r & 0 & -\tan z \\ 0 & -\tan z & 0 \end{array} \right| ,$$

$$\Gamma^{z}{}_{jk} = \left| \begin{array}{ccc} \sin z \cos z & 0 & 0 \\ 0 & \sin z \ \cos z \sin^2 r & 0 \\ 0 & 0 & 0 \end{array} \right| ,$$

$$\gamma_{122} = \frac{1}{\cos z \tan r} \; , \qquad \gamma_{311} = -\tan z \; , \qquad \gamma_{322} = -\tan z \; . \tag{1.5}$$

So, general covariant Duffin–Kemmer equation (1.1) takes the form

$$\Big\{ i\beta^0 \frac{\partial}{\partial t} + \frac{1}{\cos z} \left(i\beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{i\partial_\phi - eB(\cos r - 1) + iJ^{12}\cos r}{\sin r} \right)$$
$$+ i\beta^3 \frac{\partial}{\partial z} + i\frac{\sin z}{\cos z} \, (\beta^1 J^{13} + \beta^2 J^{23}) - M \Big\} \Psi = 0 \; , \tag{1.6}$$

In the limit of flat Minkowaki space, eq. (1.6) becomes simpler

$$\Big\{ i\beta^0 \frac{\partial}{\partial t} + i\beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{i\partial_\phi + eBr^2/2 + iJ^{12}}{r} + i\beta^3 \frac{\partial}{\partial z} - M \Big\} \Psi = 0 \; . \tag{1.7}$$

To separate the variable we will need an explicit representation for Duffin–Kemmer matrices β^a; most convenient for us is the cyclic representation; in particular, then J^{12} is diagonal (we will use blocks structure in accordance with the structure $1-3-3-3$) :

$$\beta^0 = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| , \qquad \beta^i = \left| \begin{array}{cccc} 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & \tau_i \\ -e_i^+ & 0 & 0 & 0 \\ 0 & -\tau_i & 0 & 0 \end{array} \right| , \tag{1.8}$$

where $e_i, \; e_i^t, \; \tau_i$ designate

$$e_1 = \frac{1}{\sqrt{2}}(-i, \; 0, \; i) \; , \qquad e_2 = \frac{1}{\sqrt{2}}(1, \; 0, \; 1) \; , \qquad e_3 = (0, i, 0) \; ,$$

$$\tau_1 = \frac{1}{\sqrt{2}} \left| \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right| , \tau_2 = \frac{1}{\sqrt{2}} \left| \begin{array}{ccc} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{array} \right| , \tau_3 = \left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right| = s_3 \; . \tag{1.9}$$

Entering eq. (1.6), the matrix J^{12} is

$$J^{12} = \beta^1 \beta^2 - \beta^2 \beta^1 =$$

$$
= \begin{vmatrix} -e_1 e_2^+ + e_2 e_1^+ & 0 & 0 & 0 \\ 0 & -\tau_1\tau_2 + \tau_2\tau_1 & 0 & 0 \\ 0 & 0 & -e_1^+ \bullet e_2 + e_2^+ \bullet e_1 & 0 \\ 0 & 0 & 0 & -\tau_1\tau_2 + \tau_2\tau_1 \end{vmatrix} =
$$

$$
= -i \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_3 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_3 \end{vmatrix} = -iS_3. \tag{1.8}
$$

2. Separation of the variables

Let us rewrite eq. (1.6) in the form

$$
\Big[i\beta^0 \cos z \frac{\partial}{\partial t} + i\beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{i\partial_\phi - eB(\cos r - 1) + iJ^{12}\cos r}{\sin r}
$$

$$
+ i\beta^3 \cos z \frac{\partial}{\partial z} + i \sin z (\beta^1 J^{13} + \beta^2 J^{23}) - \cos z M \Big] \Psi = 0 \; . \tag{2.1}
$$

To separate the variables, we will use the following substitution for the wave function

$$
\Psi = e^{-i\epsilon t} e^{im\phi} \begin{vmatrix} \Phi_0(r,z) \\ \vec{\Phi}(r,z) \\ \vec{E}(r,z) \\ \vec{H}(r,z) \end{vmatrix} . \tag{2.2}
$$

Eq. (2.1) leads us to (let $m + B(1 - \cos r) = \nu(r)$)

$$
\Big\{ \epsilon \cos z \; \beta^0 + i\beta^1 \; \frac{\partial}{\partial r} - \beta^2 \; \frac{\nu(r) - \cos r \; S_3}{\sin r}
$$

$$
+ i\beta^3 \; \cos z \; \frac{\partial}{\partial z} + i \; (\beta^1 J^{13} + \beta^2 J^{23}) \; \sin z - \cos z \; M \Big\} \begin{vmatrix} \Phi_0(r,z) \\ \vec{\Phi}(r,z) \\ \vec{E}(r,z) \\ \vec{H}(r,z) \end{vmatrix} = 0 \; , \tag{2.3}
$$

With the help of auxiliary relations

$$
J^{13} = \beta^1\beta^3 - \beta^3\beta^1 =
$$

$$
= \left| \begin{array}{cccc} -e_1 e_3^+ + e_3 e_1^+ & 0 & 0 & 0 \\ 0 & -\tau_1\tau_3 + \tau_3\tau_1 & 0 & 0 \\ 0 & 0 & -e_1^+ \bullet e_3 + e_3^+ \bullet e_1 & 0 \\ 0 & 0 & 0 & -\tau_1\tau_3 + \tau_3\tau_1 \end{array} \right| =
$$

$$
= i \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & \tau_2 & 0 & 0 \\ 0 & 0 & \tau_2 & 0 \\ 0 & 0 & 0 & \tau_2 \end{array} \right| = i S_2 \; ,
$$

$$
J^{23} = \beta^2 \beta^3 - \beta^3 \beta^2 =
$$

$$
= \left| \begin{array}{cccc} -e_2 e_3^+ + e_3 e_2^+ & 0 & 0 & 0 \\ 0 & -\tau_2\tau_3 + \tau_3\tau_2 & 0 & 0 \\ 0 & 0 & -e_2^+ \bullet e_3 + e_3^+ \bullet e_2 & 0 \\ 0 & 0 & 0 & -\tau_2\tau_3 + \tau_3\tau_2 \end{array} \right| =
$$

$$
= -i \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & 0 \\ 0 & 0 & \tau_1 & 0 \\ 0 & 0 & 0 & \tau_1 \end{array} \right| = -i S_1 \; ,
$$

we get

$$
(\beta^1 J^{13} + \beta^2 J^{23}) =
$$

$$
= i \left| \begin{array}{cccc} 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & \tau_1 \\ -e_1^+ & 0 & 0 & 0 \\ 0 & -\tau_1 & 0 & 0 \end{array} \right| \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & \tau_2 & 0 & 0 \\ 0 & 0 & \tau_2 & 0 \\ 0 & 0 & 0 & \tau_2 \end{array} \right| -
$$

$$
-i \left| \begin{array}{cccc} 0 & 0 & e_2 & 0 \\ 0 & 0 & 0 & \tau_2 \\ -e_2^+ & 0 & 0 & 0 \\ 0 & -\tau_2 & 0 & 0 \end{array} \right| \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & 0 \\ 0 & 0 & \tau_1 & 0 \\ 0 & 0 & 0 & \tau_1 \end{array} \right|
$$

$$
= i \left| \begin{array}{cccc} 0 & 0 & e_1\tau_2 - e_2\tau_1 & 0 \\ 0 & 0 & 0 & \tau_1\tau_2 - \tau_2\tau_1 \\ 0 & 0 & 0 & 0 \\ 0 & -\tau_1\tau_2 + \tau_2\tau_1 & 0 & 0 \end{array} \right| = \left| \begin{array}{cccc} 0 & 0 & -2e_3 & 0 \\ 0 & 0 & 0 & -\tau_3 \\ 0 & 0 & 0 & 0 \\ 0 & +\tau_3 & 0 & 0 \end{array} \right|
$$

eq. (2.3) can be presented as

$$
\Bigg[\epsilon \; \cos z \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| + i \left| \begin{array}{cccc} 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & \tau_1 \\ -e_1^+ & 0 & 0 & 0 \\ 0 & -\tau_1 & 0 & 0 \end{array} \right| \frac{\partial}{\partial r}
$$

$$
-\frac{1}{\sin r}\begin{vmatrix} 0 & 0 & e_2 & 0 \\ 0 & 0 & 0 & \tau_2 \\ -e_2^{+} & 0 & 0 & 0 \\ 0 & -\tau_2 & 0 & 0 \end{vmatrix}(\nu - \cos r\ S_3)
$$

$$
+i\cos z\begin{vmatrix} 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & \tau_3 \\ -e_3^{+} & 0 & 0 & 0 \\ 0 & -\tau_3 & 0 & 0 \end{vmatrix}\frac{\partial}{\partial z}
$$

$$
+i\sin z\begin{vmatrix} 0 & 0 & -2e_3 & 0 \\ 0 & 0 & 0 & -\tau_3 \\ 0 & 0 & 0 & 0 \\ 0 & +\tau_3 & 0 & 0 \end{vmatrix} - M\ \cos z \Bigg]\begin{vmatrix} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{vmatrix} = 0\ . \tag{2.4}
$$

In block form it is written

$$
ie_1\partial_r\vec{E} - \frac{1}{\sin r}\ e_2(\nu - \cos r\ s_3)\vec{E} + i(\cos z\ \ \partial_z - 2\sin z)e_3\vec{E} = M\cos z\ \Phi_0\ ,
$$

$$
i\epsilon\cos z\,\vec{E} + i\tau_1\partial_r\vec{H} - \frac{\tau_2}{\sin r}(\nu - \cos r\ s_3)\vec{H} + i(\cos z\ \partial_z - \sin z)\tau_3\vec{H} = M\cos z\vec{\Phi},
$$

$$
-i\epsilon\cos z\,\vec{\Phi} - ie_1^{+}\partial_r\Phi_0 + \frac{\nu}{\sin r}\ e_2^{+}\Phi_0 - i\cos z\ e_3^{+}\partial_z\Phi_0 = M\cos z\ \vec{E}\ ,
$$

$$
-i\tau_1\partial_r\vec{\Phi} + \frac{(\nu - \cos r\ s_3)}{\sin r}\ \tau_2\vec{\Phi} - i(\cos z\ \partial_z - i\sin z)\tau_3\vec{\Phi} = M\cos z\ \vec{H}\ . \tag{2.5}
$$

After simple calculation, we arrive at a system of 10 equations (let $\gamma = 1/\sqrt{2}$)

$$
\gamma(\frac{\partial E_1}{\partial r} - \frac{\partial E_3}{\partial r}) - \frac{\gamma}{\sin r}\ [(\nu - \cos r)E_1 + (\nu + \cos r)E_3] -
$$

$$
-(\cos z\frac{\partial}{\partial z} - 2\sin z)E_2 = M\cos z\Phi_0\ ,
$$

$$
+i\epsilon\cos zE_1 + i\gamma\frac{\partial H_2}{\partial r} + i\gamma\frac{\nu}{\sin r}H_2 + i(\cos z\frac{\partial}{\partial z} - \sin z)H_1 = M\cos z\Phi_1\ ,
$$

$$
+i\epsilon\cos zE_2 + i\gamma(\frac{\partial H_1}{\partial r} + \frac{\partial H_3}{\partial r}) - \frac{i\gamma}{\sin r}[(\nu - \cos r)H_1 -
$$

$$
-(\nu + \cos r)H_3] = M\cos z\Phi_2\ ,
$$

$$+i\epsilon\cos zE_3 + i\gamma\ \frac{\partial H_2}{\partial r} - i\gamma\frac{\nu}{\sin r}H_2 - i(\cos z\frac{\partial}{\partial z} - \sin z)H_3 = M\cos z\Phi_3 \tag{2.6}$$

$$\begin{gathered}
-i\epsilon\cos z\Phi_1 + \gamma\frac{\partial\Phi_0}{dr} + \gamma\frac{\nu}{\sin r}\Phi_0 = M\cos zE_1\ ,\\
-i\epsilon\cos z\Phi_2 - \cos z\frac{\partial\Phi_0}{\partial z} = M\cos zE_2\ ,\\
-i\epsilon\cos z\Phi_3 - \gamma\frac{\partial\Phi_0}{\partial r} + \gamma\frac{\nu}{\sin r}\Phi_0 = M\cos zE_3\ ,
\end{gathered} \tag{2.7}$$

$$\begin{gathered}
-i\gamma\frac{\partial\Phi_2}{\partial r} - i\gamma\frac{\nu}{\sin r}\Phi_2 - i(\cos z\frac{\partial}{\partial z} - \sin z)\Phi_1 = M\cos zH_1\ ,\\
-i\gamma(\frac{\partial\Phi_1}{\partial r} + \frac{\partial\Phi_3}{\partial r}) + \frac{i\gamma}{\sin r}[(\nu - \cos r)\Phi_1 - (\nu + \cos r)\Phi_3] = M\cos zH_2\ ,\\
-i\gamma\frac{\partial\Phi_2}{\partial r} + i\gamma\frac{\nu}{\sin r}\Phi_2 + i(\cos z\frac{\partial}{\partial z} - \sin z)\Phi_3 = M\cos zH_3\ .
\end{gathered} \tag{2.8}$$

With the help of substitutions

$$\begin{gathered}
H_1 = \frac{h_1}{\cos z}\ , \qquad (\cos z\ \frac{\partial}{\partial z} - \sin z)H_1 = \frac{\partial h_1}{\partial z}\ ,\\
H_3 = \frac{h_3}{\cos z}\ , \qquad (\cos z\ \frac{\partial}{\partial z} - \sin z)H_3 = \frac{\partial h_3}{\partial z}\ ,\\
\Phi_1 = \frac{\varphi_1}{\cos z}\ , \qquad (\cos z\ \frac{\partial}{\partial z} - \sin z)\Phi_1 = \frac{\partial\varphi_1}{\partial z}\ ,\\
\Phi_3 = \frac{\varphi_3}{\cos z}\ , \qquad (\cos z\ \frac{\partial}{\partial z} - \sin z)\Phi_3 = \frac{\partial\varphi_3}{\partial z}\ ,\\
E_2 = \frac{e_2}{\cos^2 z}\ , \qquad (\cos z\ \frac{\partial}{\partial z} - 2\sin z)E_2 = \frac{1}{\cos z}\frac{\partial e_2}{\partial z}\ ,\\
E_1 = \frac{e_1}{\cos z}\ , \qquad E_3 = \frac{e_3}{\cos z}\ ,\\
\Phi_0 = \frac{\varphi_0}{\cos^2 z}\ , \qquad \Phi_2 = \frac{\varphi_2}{\cos^2 z}\ , \qquad H_2 = \frac{h_2}{\cos^2 z}\ ,
\end{gathered} \tag{2.9}$$

we get a more simple system

$$\gamma\,(\frac{\partial e_1}{\partial r}-\frac{\partial e_3}{\partial r})-\frac{\gamma}{\sin r}\,[(\nu-\cos r)e_1+(\nu+\cos r)e_3]-\frac{\partial e_2}{\partial z}=M\varphi_0\ ,$$
$$+i\epsilon e_1+i\frac{\gamma}{\cos^2 z}(\frac{\partial}{\partial r}+\frac{\nu}{\sin r})\ h_2+i\frac{\partial h_1}{\partial z}=M\varphi_1\ ,$$
$$+i\epsilon e_2+i\gamma(\frac{\partial h_1}{\partial r}+\frac{\partial h_3}{\partial r})-\frac{i\gamma}{\sin r}\,[(\nu-\cos r)h_1-(\nu+\cos r)h_3]=M\varphi_2\ ,$$
$$+i\epsilon e_3+i\frac{\gamma}{\cos^2 z}(\frac{\partial}{\partial r}-\frac{\nu}{\sin r})h_2-i\frac{\partial h_3}{\partial z}=M\varphi_3\ . \tag{2.10}$$

$$-i\epsilon\varphi_1+\frac{\gamma}{\cos^2 z}(\frac{\partial}{\partial r}+\frac{\nu}{\sin r})\varphi_0=Me_1\ ,$$
$$-i\epsilon\varphi_2-(\frac{\partial}{\partial z}+2\frac{\sin z}{\cos z})\varphi_0=Me_2\ ,$$
$$-\ i\epsilon\varphi_3-\frac{\gamma}{\cos^2 z}(\frac{\partial}{\partial r}-\frac{\nu}{\sin r})\ \varphi_0=Me_3\ , \tag{2.11}$$

$$-i\frac{\gamma}{\cos^2 z}\ (\frac{\partial}{dr}+\frac{\nu}{\sin r})\varphi_2\ -i\frac{\partial\varphi_1}{\partial z}=Mh_1\ ,$$
$$-i\gamma(\frac{\partial\varphi_1}{dr}+\frac{d\varphi_3}{dr})\ +\frac{i\gamma}{\sin r}[(\nu-\cos r)\varphi_1-(\nu+\cos r)\varphi_3]=Mh_2\ ,$$
$$-i\frac{\gamma}{\cos^2 z}(\frac{\partial}{\partial r}-\frac{\nu}{\sin r})\varphi_2+i\frac{\partial\varphi_3}{\partial z}=Mh_3\ . \tag{2.12}$$

These equation can be transformed to the form

$$\gamma\,(\frac{\partial}{\partial r}-\frac{\nu-\cos r}{\sin r})\ e_1-\gamma(\frac{\partial}{\partial r}+\frac{\nu+\cos r}{\sin r})\ e_3-\frac{\partial e_2}{\partial z}=M\ \varphi_0\ ,$$
$$i\gamma(\frac{\partial}{\partial r}-\frac{\nu-\cos r}{\sin r})\ h_1+i\gamma(\frac{\partial}{\partial r}+\frac{\nu+\cos r}{\sin r})\ h_3+i\epsilon\ e_2=M\ \varphi_2\ ,$$
$$\frac{i\gamma}{\cos^2 z}(\frac{\partial}{\partial r}+\frac{\nu}{\sin r})\ h_2+i\epsilon\ e_1+i\frac{\partial h_1}{\partial z}=M\ \varphi_1\ ,$$
$$\frac{i\gamma}{\cos^2 z}(\frac{\partial}{\partial r}-\frac{\nu}{\sin r})h_2+i\epsilon\ e_3-i\frac{\partial h_3}{\partial z}=M\ \varphi_3\ ,$$

$$(2.13a)$$

$$\frac{\gamma}{\cos^2 z}\;(\frac{\partial}{\partial r}+\frac{\nu}{\sin r})\;\varphi_0-i\epsilon\;\varphi_1=M\;e_1\;,$$
$$-\frac{i\gamma}{\cos^2 z}\;(\frac{\partial}{dr}+\frac{\nu}{\sin r})\varphi_2\;-i\frac{\partial\varphi_1}{\partial z}=M\;h_1\;,$$
$$-\frac{\gamma}{\cos^2 z}(\frac{\partial}{\partial r}-\frac{\nu}{\sin r})\;\varphi_0-i\epsilon\;\varphi_3=M\;e_3\;,$$
$$-\frac{i\gamma}{\cos^2 z}(\frac{\partial}{\partial r}-\frac{\nu}{\sin r})\;\varphi_2+i\frac{\partial\varphi_3}{\partial z}=M\;h_3\;,$$
$$-i\epsilon\varphi_2-(\frac{\partial}{\partial z}+2\frac{\sin z}{\cos z})\varphi_0=M\;e_2\;,$$
$$-i\gamma(\frac{\partial}{\partial r}-\frac{\nu-\cos r}{\sin r})\;\varphi_1-i\gamma(\frac{\partial}{\partial r}\;+\frac{\nu+\cos r}{\sin r})\;\varphi_3=M\;h_2\;. \qquad (2.13b)$$

Let us introduce a shortened notation

$$\gamma(\frac{\partial}{\partial r}+\frac{\nu-\cos r}{\sin r})=\hat{a}_-,\gamma(\frac{\partial}{\partial r}+\frac{\nu+\cos r}{\sin r})=\hat{a}_+,\gamma(\frac{\partial}{\partial r}+\frac{\nu}{\sin r})=\hat{a},$$
$$\gamma(-\frac{\partial}{\partial r}+\frac{\nu-\cos r}{\sin r})=\hat{b}_-,\gamma(-\frac{\partial}{\partial r}+\frac{\nu+\cos r}{\sin r})=\hat{b}_+,\gamma(-\frac{\partial}{\partial r}+\frac{\nu}{\sin r})=\hat{b}\;, \qquad (2.14)$$

then the above equations read

$$-\hat{b}_-\;e_1-\hat{a}_+\;e_3-\frac{\partial e_2}{\partial z}=M\;\varphi_0\;,$$
$$-i\hat{b}_-\;h_1+i\hat{a}_+\;h_3+i\epsilon\;e_2=M\;\varphi_2\;,$$
$$\frac{i}{\cos^2 z}\hat{a}\;h_2+i\epsilon\;e_1+i\frac{\partial h_1}{\partial z}=M\;\varphi_1\;,$$
$$-\frac{i}{\cos^2 z}\hat{b}\;h_2+i\epsilon\;e_3-i\frac{\partial h_3}{\partial z}=M\;\varphi_3\;, \qquad (2.15)$$
$$\frac{1}{\cos^2 z}\;\hat{a}\;\varphi_0-i\epsilon\;\varphi_1=M\;e_1\;,$$
$$-\frac{i}{\cos^2 z}\;\hat{a}\;\varphi_2\;-i\frac{\partial\varphi_1}{\partial z}=M\;h_1\;,$$

$$\frac{1}{\cos^2 z}\ \hat{b}\ \varphi_0 - i\epsilon\ \varphi_3 = M\ e_3\ ,$$
$$\frac{i}{\cos^2 z}\ \hat{b}\ \varphi_2 + i\frac{\partial\varphi_3}{\partial z} = M\ h_3\ .$$
$$-i\epsilon\varphi_2 - (\frac{\partial}{\partial z} + 2\frac{\sin z}{\cos z})\varphi_0 = M\ e_2\ ,$$
$$i\hat{b}_-\ \varphi_1 - i\hat{a}_+\ \varphi_3 = M\ h_2\ , \tag{2.16}$$

We can note that turning back to Φ_0, we get a simple system as well

$$-\hat{b}_-\ e_1 - \hat{a}_+\ e_3 - \frac{\partial e_2}{\partial z} = M\ \cos^2 z\Phi_0\ ,$$
$$-i\hat{b}_-\ h_1 + i\hat{a}_+\ h_3 + i\epsilon\ e_2 = M\ \varphi_2\ ,$$
$$\frac{i}{\cos^2 z}\hat{a}\ h_2 + i\epsilon\ e_1 + i\frac{\partial h_1}{\partial z} = M\ \varphi_1\ ,$$
$$-\frac{i}{\cos^2 z}\hat{b}\ h_2 + i\epsilon\ e_3 - i\frac{\partial h_3}{\partial z} = M\ \varphi_3\ , \tag{2.17}$$

$$\hat{a}\ \Phi_0 - i\epsilon\ \varphi_1 = M\ e_1\ ,$$
$$-\frac{i}{\cos^2 z}\ \hat{a}\ \varphi_2\ - i\frac{\partial\varphi_1}{\partial z} = M\ h_1\ ,$$
$$\hat{b}\ \Phi_0 - i\epsilon\ \varphi_3 = M\ e_3\ ,$$
$$\frac{i}{\cos^2 z}\ \hat{b}\ \varphi_2 + i\frac{\partial\varphi_3}{\partial z} = M\ h_3\ .$$
$$-i\epsilon\ \varphi_2 - \cos^2 z\ \frac{\partial\Phi_0}{\partial z} = M\ e_2\ ,$$
$$i\hat{b}_-\ \varphi_1 - i\hat{a}_+\ \varphi_3 = M\ h_2\ . \tag{2.18}$$

Below we will work with equations (2.17) – (2.18).

3. Transition to a non-relativistic approximation

Excluding from (2.17)–(2.18) non-dynamical variables Φ_0, h_1, h_2, h_3:

$$\frac{1}{\cos^2 z}(-\hat{b}_- \; e_1 - \hat{a}_+ \; e_3 - \frac{\partial e_2}{\partial z}) = M \; \Phi_0 \; ,$$

$$-\frac{i}{\cos^2 z} \; \hat{a} \; \varphi_2 \; - i\frac{\partial \varphi_1}{\partial z} = M \; h_1 \; ,$$

$$i\hat{b}_- \; \varphi_1 - i\hat{a}_+ \; \varphi_3 = M \; h_2 \; ,$$

$$\frac{i}{\cos^2 z} \; \hat{b} \; \varphi_2 + i\frac{\partial \varphi_3}{\partial z} = M \; h_3 \; . \tag{3.1}$$

we obtain 6 equations (grouping them in pair)

$$\frac{i}{\cos^2 z}\hat{a} \; (i\hat{b}_- \; \varphi_1 - i\hat{a}_+ \; \varphi_3) + i\epsilon \; Me_1 + i\frac{\partial}{\partial z}(-\frac{i}{\cos^2 z} \; \hat{a} \; \varphi_2 \; - i\frac{\partial \varphi_1}{\partial z}) = M^2 \; \varphi_1 \; ,$$

$$\hat{a} \; \frac{1}{\cos^2 z}(-\hat{b}_- \; e_1 - \hat{a}_+ \; e_3 - \frac{\partial e_2}{\partial z}) - i\epsilon \; M\varphi_1 = M^2 \; e_1 \; , \tag{3.2a}$$

$$-i\hat{b}_- \; (-\frac{i}{\cos^2 z} \; \hat{a} \; \varphi_2 \; - i\frac{\partial \varphi_1}{\partial z}) + i\hat{a}_+ \; (\frac{i}{\cos^2 z} \; \hat{b} \; \varphi_2 + i\frac{\partial \varphi_3}{\partial z}) + i\epsilon \; Me_2 = M^2 \; \varphi_2 \; ,$$

$$-i\epsilon \; M\varphi_2 - \cos^2 z \; \frac{\partial}{\partial z}\frac{1}{\cos^2 z}(-\hat{b}_- \; e_1 - \hat{a}_+ \; e_3 - \frac{\partial e_2}{\partial z}) = M^2 \; e_2 \; , \tag{3.2b}$$

$$-\frac{i}{\cos^2 z}\hat{b} \; (i\hat{b}_- \; \varphi_1 - i\hat{a}_+ \; \varphi_3) + i\epsilon \; Me_3 - i\frac{\partial}{\partial z}(\frac{i}{\cos^2 z} \; \hat{b} \; \varphi_2 + i\frac{\partial \varphi_3}{\partial z}) = M^2 \; \varphi_3 \; ,$$

$$\hat{b} \; \frac{1}{\cos^2 z}(-\hat{b}_- \; e_1 - \hat{a}_+ \; e_3 - \frac{\partial e_2}{\partial z}) - i\epsilon \; M\varphi_3 = M^2 \; e_3 \; . \tag{3.2c}$$

Now we should introduce big Ψ_i and small ψ_i components

$$\varphi_1 = \Psi_1 + \psi_1 \; , \qquad ie_1 = \Psi_1 - \psi_1 \; ,$$

$$\varphi_2 = \Psi_2 + \psi_2 \; , \qquad ie_2 = \Psi_2 - \psi_2 \; ,$$
$$\varphi_3 = \Psi_3 + \psi_3 \; , \qquad ie_3 = \Psi_3 - \psi_3 \; ,$$

and in the same time separate the rest energy by formal change $\epsilon \Longrightarrow (\epsilon + M)$ – so we arrive at

$$
\begin{gathered}
-\frac{\hat{a}\hat{b}_-}{\cos^2 z}(\Psi_1 + \psi_1) + \frac{\hat{a}\hat{a}_+}{\cos^2 z}(\Psi_3 + \psi_3) + \frac{\hat{a}}{\cos^2 z}(\frac{\partial}{\partial z} + \frac{2\sin z}{\cos z})(\Psi_2 + \psi_2) \\
+\frac{\partial^2}{\partial z^2}(\Psi_1 + \psi_1) + (\epsilon + M)M(\Psi_1 - \psi_1) = M^2(\Psi_1 + \psi_1) \; , \\
-\frac{\hat{a}\hat{b}_-}{\cos^2 z}(\Psi_1 - \psi_1) - \frac{\hat{a}\hat{a}_+}{\cos^2 z}(\Psi_3 - \psi_3) - \frac{\hat{a}}{\cos^2 z}\frac{\partial}{\partial z}(\Psi_2 - \psi_2) \\
+(\epsilon + M)M(\Psi_1 + \psi_1) = M^2(\Psi_1 - \psi_1) \; ;
\end{gathered}
\tag{3.3a}
$$

$$
\begin{gathered}
-\frac{\hat{b}_-\hat{a} + \hat{a}_+\hat{b}}{\cos^2 z}(\Psi_2 + \psi_2) - \hat{b}_-\frac{\partial}{\partial z}(\Psi_1 + \psi_1) - \hat{a}_+\frac{\partial}{\partial z}(\Psi_3 + \psi_3) \\
+(\epsilon + M)M(\Psi_2 - \psi_2) = M^2(\Psi_2 + \psi_2) \; , \\
\hat{b}_- \cos^2 z \; \frac{\partial}{\partial z}\frac{1}{\cos^2 z}(\Psi_1 - \psi_1) + \hat{a}_+ \cos^2 z \; \frac{\partial}{\partial z}\frac{1}{\cos^2 z}(\Psi_3 - \psi_3) \\
+\cos^2 z \; \frac{\partial}{\partial z}\frac{1}{\cos^2 z}\frac{\partial}{\partial z}(\Psi_2 - \psi_2) + (\epsilon + M)M(\Psi_2 + \psi_2) = M^2(\Psi_2 - \psi_2) \; ;
\end{gathered}
\tag{3.3b}
$$

$$
\begin{gathered}
\frac{\hat{b}\hat{b}_-}{\cos^2 z}(\Psi_1 + \psi_1) - \frac{\hat{b}\hat{a}_+}{\cos^2 z}(\Psi_3 + \psi_3) + \frac{1}{\cos^2 z}(\frac{\partial}{\partial z} + \frac{2\sin z}{\cos z})\hat{b}(\Psi_2 + \psi_2) + \\
+\frac{\partial^2}{\partial z^2}(\Psi_3 + \psi_3) + (\epsilon + M)M(\Psi_3 - \psi_3) = M^2(\Psi_3 + \psi_3) \; , \\
-\frac{\hat{b}\hat{b}_-}{\cos^2 z}(\Psi_1 - \psi_1) - \frac{\hat{b}\hat{a}_+}{\cos^2 z}(\Psi_3 - \psi_3) - \frac{\hat{b}}{\cos^2 z}\frac{\partial}{\partial z}(\Psi_2 - \psi_2) \\
+(\epsilon + M)M(\Psi_3 + \psi_3) = M^2(\Psi_3 - \psi_3) \; .
\end{gathered}
\tag{3.3c}
$$

Summing equation for each pair and neglecting small components ψ_k in comparison with big ones Ψ_k, we get

$$\left(-\frac{2}{\cos^2 z}\,\hat{a}\hat{b}_- + \frac{\partial^2}{\partial z^2} + 2\epsilon M\right)\Psi_1 + 2\frac{\sin z}{\cos^3 z}\,\hat{a}\Psi_2 = 0\,,$$

$$\left(-\frac{2}{\cos^2 z}\,\hat{b}\hat{a}_+ + \frac{\partial^2}{\partial z^2} + 2\epsilon M\right)\Psi_3 + 2\frac{\sin z}{\cos^3 z}\,\hat{b}\Psi_2 = 0\,,$$

$$\left(-\frac{1}{\cos^2 z}(\hat{b}_-\hat{a} + \hat{a}_+\hat{b}) + 2\epsilon M + \frac{\partial^2}{\partial z^2} + 2\frac{\sin z}{\cos z}\frac{\partial}{\partial z}\right)\Psi_2$$

$$+2\frac{\sin z}{\cos z}(\hat{b}_-\Psi_1 + \hat{a}_+\Psi_3) = 0\,. \qquad (3.4a)$$

It is a needed system in Pauli approximation. In particular, for the case of flat space model we get much more simple system of three separated equations

$$\left(-2\hat{a}\hat{b}_- + \frac{\partial^2}{\partial z^2} + 2\epsilon M\right)\Psi_1 = 0\,,$$

$$\left(-2\hat{b}\hat{a}_+ + \frac{\partial^2}{\partial z^2} + 2\epsilon M\right)\Psi_3 = 0\,,$$

$$\left(-(\hat{b}_-\hat{a} + \hat{a}_+\hat{b}) + 2\epsilon M + \frac{\partial^2}{\partial z^2}\right)\ \Psi_2 = 0\,,$$

where in definitions for $\hat{a}, \hat{b}, \hat{a}_-, \hat{b}_-, \hat{a}_+, \hat{b}_+$ some simplifications are to be performed – see (2.14).

Equations (3.4a) can be transformed to a more symmetrical form if one make a substitution

$$\Psi_2 = \cos z\bar{\Psi}_2\,,$$

$$\left(\frac{\partial^2}{\partial z^2} + 2\frac{\sin z}{\cos z}\frac{\partial}{\partial z}\right)\cos z\bar{\Psi}_2 = \cos z\left(\frac{\partial^2}{\partial z^2} - \frac{2}{\cos^2 z} + 1\right)\bar{\Psi}_2\,, \qquad (3.4b)$$

Then, eqs. (3.4a) read

$$\left(-\frac{2\hat{a}\hat{b}_-}{\cos^2 z} + \frac{\partial^2}{\partial z^2} + 2\epsilon M\right)\Psi_1 + 2\frac{\sin z}{\cos^2 z}\,\hat{a}\bar{\Psi}_2 = 0\,,$$

$$\left(-\frac{2\hat{b}\hat{a}_+}{\cos^2 z} + \frac{\partial^2}{\partial z^2} + 2\epsilon M\right)\Psi_3 + 2\frac{\sin z}{\cos^2 z}\,\hat{b}\bar{\Psi}_2 = 0\,,$$

$$\left(-\frac{(\hat{b}_-\hat{a} + \hat{a}_+\hat{b} + 2)}{\cos^2 z} + \frac{\partial^2}{\partial z^2} + 2\epsilon M + 1\right)\bar{\Psi}_2$$

$$+2\frac{\sin z}{\cos^2 z}(\hat{b}_-\Psi_1 + \hat{a}_+\Psi_3) = 0 \; . \qquad (3.4c)$$

Let us introduce new functions

$$\hat{b}_-\Psi_1 = G_1 \; , \qquad \bar{\Psi}_2 = G_2 \; , \qquad \hat{a}_+\Psi_3 = G_3 \; , \qquad (3.5a)$$

eqs. (3.4a) will give

$$\left(-\frac{2}{\cos^2 z}\,\hat{b}_-\hat{a} + \frac{\partial^2}{\partial z^2} + 2\epsilon M\right) G_1 + 2\frac{\sin z}{\cos^2 z}\,\hat{b}_-\hat{a}\,G_2 = 0 \; ,$$

$$\left(-\frac{2}{\cos^2 z}\,\hat{a}_+\hat{b} + \frac{\partial^2}{\partial z^2} + 2\epsilon M\right) G_3 + 2\frac{\sin z}{\cos^2 z}\,\hat{a}_+\hat{b}\,G_2 = 0 \; ,$$

$$\left(-\frac{1}{\cos^2 z}(\hat{b}_-\hat{a} + \hat{a}_+\hat{b} + 2) + \frac{\partial^2}{\partial z^2} + 2\epsilon M + 1\right) G_2$$

$$+2\frac{\sin z}{\cos^2 z}\,(G_1 + G_3) = 0 \; . \qquad (3.5b)$$

Now we should define a factorized form for three functions

$$G_1 = Z_1(z)R_1(r) \; , \; G_2 = Z_2(z)R_2(r) \; , \; G_3 = Z_3(z)R_3(r) \; ; \qquad (3.6a)$$

then eqs. (3.5b) read

$$\left(-\frac{2}{\cos^2 z}\,\hat{b}_-\hat{a} + \frac{\partial^2}{\partial z^2} + 2\epsilon M\right) Z_1R_1 + 2\frac{\sin z}{\cos^2 z}\,\hat{b}_-\hat{a}\,Z_2R_2 = 0 \; ,$$

$$\left(-\frac{2}{\cos^2 z}\,\hat{a}_+\hat{b} + \frac{\partial^2}{\partial z^2} + 2\epsilon M\right) Z_3R_3 + 2\frac{\sin z}{\cos^2 z}\,\hat{a}_+\hat{b}\,Z_2R_2 = 0 \; ,$$

$$\left(-\frac{1}{\cos^2 z}(\hat{b}_-\hat{a} + \hat{a}_+\hat{b} + 2) + \frac{\partial^2}{\partial z^2} + 2\epsilon M + 1\right)\; Z_2R_2$$

$$+2\frac{\sin z}{\cos^2 z}\,(Z_1R_1 + Z_3R_3) = 0 \; . \qquad (3.6b)$$

Note that the first equation in (3.6b) does not change if one acts from the left by the operator $\hat{b}_-\hat{a}$; similarly the second equation preserves its form if on acts from the left by the operator $\hat{a}_+\hat{b}$. Therefore, one cam assume existence of the following radial relationships

$$\hat{b}_-\hat{a}R_1 = \lambda\, R_1 \; , \qquad \hat{b}_-\hat{a}R_2 = \lambda\, R_2 \; , \qquad R_1 = R_2 = R \; ; \qquad (3.7a)$$

and

$$\hat{a}_+\hat{b}R_3 = \lambda' \; R_3 \; , \qquad \hat{a}_+\hat{b}R_2 = \lambda' \; R_2 \; , \qquad R_2 = R_3 = R \; . \tag{3.7b}$$

Taking into account these restrictions from (3.6b) we obtain the system in z variable

$$\begin{gathered}
\left(-\frac{2\lambda}{\cos^2 z} \; + \frac{d^2}{dz^2} + 2\epsilon M\right) Z_1 + 2\lambda \frac{\sin z}{\cos^2 z} \; Z_2 = 0 \; , \\
\left(-\frac{2\lambda'}{\cos^2 z} + \frac{d^2}{dz^2} + 2\epsilon M\right) Z_3 + 2\lambda' \frac{\sin z}{\cos^2 z} \; Z_2 = 0 \; , \\
\left(-\frac{1}{\cos^2 z}(\lambda + \lambda' + 2) + \frac{d^2}{dz^2} + 2\epsilon M + 1\right) \; Z_2 + 2\frac{\sin z}{\cos^2 z} \; (Z_1 + Z_3) = 0 \; .
\end{gathered} \tag{3.8}$$

With the use of explicit expressions for operators $\hat{a}, \hat{a}_+, \hat{b}, \hat{b}_-$, we derive

$$\begin{gathered}
\hat{b}_-\hat{a} = \frac{1}{2}\left(-\frac{d^2}{dr^2} - \frac{\cos r}{\sin r}\frac{d}{dr} - B + \frac{\nu^2(r)}{\sin^2 r}\right) \; , \\
\hat{a}_+\hat{b} = \frac{1}{2}\left(-\frac{d^2}{dr^2} - \frac{\cos r}{\sin r}\frac{d}{dr} + B + \frac{\nu^2(r)}{\sin^2 r}\right) \; , \\
\hat{a}_+\hat{b} = \hat{b}_-\hat{a} - B \; ,
\end{gathered}$$

so the first radial equation for R_2 takes the form

$$\begin{gathered}
\hat{b}_-\hat{a} \; R_2 = \lambda \; R_2 \qquad \Longrightarrow \\
\left(\frac{d^2}{dr^2} + \frac{\cos r}{\sin r}\frac{d}{dr} + B - \frac{\nu^2(r)}{\sin^2 r} + 2\lambda\right) R_2 = 0 \; ;
\end{gathered} \tag{3.9a}$$

the second equation for R_1 gives the same only if two parameters λ and λ' obey a special additional constraint

$$\hat{a}_+\hat{b} \; R_2 = \lambda' \; R_2 \qquad \Longrightarrow \qquad \hat{b}_-\hat{a} \; R_2 = (\lambda' + B) \; R_2,$$

that is

$$\lambda' = \lambda - B \; . \tag{3.9b}$$

Let us consider eq. (3.9a) in more detail

$$\frac{d^2}{dr^2}R + \frac{1}{\tan r}\frac{dR}{dr} - \frac{1}{\sin^2 r}[\; m + B(1 - \cos r)]^2 \; R + (B + 2\lambda) \; R = 0 \, .$$

In a new variable

$$1-\cos r = 2y\ , \qquad\qquad y=\sin^2\frac{r}{2}\in[0,\ 1\ ]\ ,$$

$$\Big[y(1-y)\frac{d^2}{dy^2}+(1-2y)\frac{d}{dy}-$$

$$-\frac{1}{4}(\frac{m^2}{y}-4B^2+\frac{(m+2B)^2}{1-y}\ )\ +(B+2\lambda)\Big]\ R=0\ . \qquad (3.10)$$

With the substitution $R=y^a(1-y)^b\ F$, eq. (3.10) gives

$$y(1-y)\ F''+[\ a(1-y)-by+a(1-y)-by+(1-2y)]\ F'$$

$$+\frac{1}{y}\ [\ a(a-1)+a-\frac{m^2}{4}\ ]\ F\ \ +\ \ \frac{1}{1-y}\ [\ b(b-1)+b-\frac{(m+2B)^2}{4}\ ]\ F$$

$$-[\ a(a+1)+2ab+b(b+1)-B^2-(B+2\lambda)\ ]\ F=0\ .$$

If parameters obey restriction below

$$a=\pm\frac{|\ m\ |}{2}\ , \qquad b=\pm\frac{|\ m+2B\ |}{2}\ ; \qquad (3.11a)$$

we arrive at a more simple equation

$$y(1-y)\ F''+[\ (2a+1)-2(a+b+1)y\ ]\ F'$$

$$-[a(a+1)+2ab+b(b+1)-B^2-(B+2\lambda)]\ F=0\ ,$$

$$(3.11b)$$

which is recognized as a hypergeometric one

$$y(1-y)\ F+[\ \gamma-(\alpha+\beta+1)y\ ]\ F'-\alpha\beta\ F=0\ . \qquad (3.11c)$$

So we have (to obtain solutions for bound states we must assume positive a and b)

$$y=\sin^2\frac{r}{2}\ , \qquad y\ \in\ [\,0,\ +1\,]\,, \qquad r\in[\,0,+\pi\,]\,,$$

$$R=(\sin\frac{r}{2})^{+|m|}\ \ (\cos\frac{r}{2})^{+|m+2B|}\ \ F(\alpha,\beta,\gamma;\ -\sin^2\frac{r}{2})\,;$$

$$(3.11d)$$

parameters $(\alpha,\beta,\gamma)$are determined by

$$\gamma=+\mid m\mid+1\,, \qquad a=+\frac{|\ m\ |}{2}\ , \qquad b=+\frac{|\ m+2B\ |}{2}\ ,$$

$$\begin{cases} \alpha + \beta = 2a + 2b + 1 \ , \\ \alpha \, \beta = (a + b)(a + b + 1) - B^2 - (B + 2\lambda) \ ; \end{cases} \tag{3.12a}$$

that is

$$\gamma = + \mid m \mid +1 \ , \ a = + \frac{\mid m \mid}{2} \ , \ b = + \frac{\mid m + 2B \mid}{2} \ ,$$

$$\alpha = a + b + \frac{1}{2} - \sqrt{\left(B + \frac{1}{2} \right)^2 + 2\lambda} \ ,$$

$$\beta = a + b + \frac{1}{2} + \sqrt{\left(B + \frac{1}{2} \right)^2 + 2\lambda} \ . \tag{3.12b}$$

To obtain solutions in polynomials, we must assume positiveness of the expression under the sign of square root and must impose restriction on the α

$$\alpha = a + b + \frac{1}{2} - \sqrt{\left(B + \frac{1}{2} \right)^2 + 2\lambda} = -n = 0, -1, -2, ..., \tag{3.13a}$$

from whence it follows the quantization rule

$$2\lambda + \left(B + \frac{1}{2} \right)^2 = (a + b + \frac{1}{2} + n)^2 > 0 \ , \tag{3.13b}$$

solutions corresponding to bound states are given by

$$R = (\sin \frac{r}{2})^{+|m|} \ \ (\cos \frac{r}{2})^{+|m+2B|}$$

$$\times \ F(-n \ , \mid m \mid + \mid m + 2B \mid +1 + n \ , \mid m \mid +1; \ -\sin^2 \frac{r}{2}) \ . \tag{3.13c}$$

Below, we will use notation

$$\lambda = \Lambda - \frac{B}{2} \tag{3.14a}$$

then the formula for spectrum (3.13b)will read

$$2\Lambda + B^2 = N(N + 1) \ , \qquad N = a + b + n \ . \tag{3.13b}$$

4. Behavior of solutions in z variable near singular points

Let us turn to the system (3.8)

$$
\begin{gathered}
\left(\frac{d^2}{dz^2} - \frac{2\lambda}{\cos^2 z} + 2\epsilon M\right) Z_1 + 2\lambda \frac{\sin z}{\cos^2 z}\, \bar{Z}_2 = 0 \, , \\
\left(\frac{d^2}{dz^2} - \frac{2\lambda'}{\cos^2 z} + 2\epsilon M\right) Z_3 + 2\lambda' \frac{\sin z}{\cos^2 z}\, \bar{Z}_2 = 0 \, , \\
\left(\frac{d^2}{dz^2} - \frac{\lambda + \lambda' + 2}{\cos^2 z} + 2\epsilon M + 1\right) \bar{Z}_2 + 2 \frac{\sin z}{\cos^2 z}\, (Z_1 + Z_3) = 0 \, .
\end{gathered} \tag{4.1b}
$$

In the variable

$$
\sin z = x \, , \qquad x \in [-1, +1] \, ,
$$

we get

$$
\begin{gathered}
\left((1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx} - \frac{2\lambda}{1-x^2} + 2\epsilon M\right) Z_1 + \frac{2\lambda x}{1-x^2}\, \bar{Z}_2 = 0 \, , \\
\left((1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx} - \frac{2\lambda'}{1-x^2} + 2\epsilon M\right) Z_3 + \frac{2\lambda' x}{1-x^2}\, \bar{Z}_2 = 0 \, , \\
\left((1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx} - \frac{2+\lambda+\lambda'}{1-x^2} + 2\epsilon M + 1\right) \bar{Z}_2 + \frac{2x}{1-x^2}\, (Z_1 + Z_3) = 0 \, .
\end{gathered} \tag{4.2}
$$

Near the point $z = +\pi/2$ we have

$$
z = +\pi/2 \, , \qquad x \to +1 \, ,
$$

$$
\begin{gathered}
\left(2(1-x)\frac{d^2}{dx^2} - \frac{d}{dx} - \frac{\lambda}{1-x}\right) Z_1 + \frac{\lambda}{1-x}\, \bar{Z}_2 = 0 \, , \\
\left(2(1-x)\frac{d^2}{dx^2} - \frac{d}{dx} - \frac{\lambda'}{1-x}\right) Z_3 + \frac{\lambda'}{1-x}\, \bar{Z}_2 = 0 \, , \\
\left(2(1-x)\frac{d^2}{dx^2} - \frac{d}{dx} - \frac{2+\lambda+\lambda'}{1-x}\right) \bar{Z}_2 + \frac{1}{1-x}\, (Z_1 + Z_3) = 0 \, ;
\end{gathered} \tag{4.3a}
$$

so the possible solution is

$$
Z_1 = A_1 (1-x)^a \, , \qquad \bar{Z}_2 = A_2 (1-x)^a \, , \qquad Z_3 = A_3 (1-x)^a \, . \tag{4.3b}
$$

Substituting (4.3b) into (4.3a), we obtain linear system with respect to A_1, A_2, A_3:

$$(2a^2 - a - \lambda)A_1 + \lambda A_2 = 0 \; ,$$

$$(2a^2 - a - \lambda')A_3 + \lambda' A_2 = 0 \; ,$$

$$(2a^2 - a - \frac{2+\lambda+\lambda'}{2})A_2 + A_1 + A_3 = 0 \; . \qquad (4.3c)$$

In similar manner consider behavior od solution near the second singular ponit

$$z = -\pi/2 \; , \qquad x \to -1 \; ,$$

$$\left(2(1+x)\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{\lambda}{1+x} \right) Z_1 - \frac{\lambda}{1+x} \; \bar{Z}_2 = 0 \; ,$$

$$\left(2(1+x)\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{\lambda'}{1+x} \right) Z_3 - \frac{\lambda'}{1+x} \; \bar{Z}_2 = 0 \; ,$$

$$\left(2(1+x)\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{2+\lambda+\lambda'}{1+x} \right) \bar{Z}_2 - \frac{1}{1+x} \; (Z_1 + Z_3) = 0 \; ; \qquad (4.4a)$$

that is

$$Z_1 = B_1 (1+x)^b \; , \qquad \bar{Z}_2 = B_2 (1+x)^b \; , \qquad Z_3 = B_3 (1+x)^b \; . \qquad (4.4b)$$

and coefficients B_1, B_2, B_3 obey the linear system as well

$$(2b^2 - b - \lambda)B_1 - \lambda B_2 = 0 \; ,$$

$$(2b^2 - b - \lambda')B_3 - \lambda' B_2 = 0 \; ,$$

$$(2b^2 - b - \frac{2+\lambda+\lambda'}{2})B_2 - B_1 - B_3 = 0 \; , \qquad (4.4c)$$

With the notation

$$2a^2 - a = A \; , \qquad 2b^2 - b = B \; ,$$

$$a = \frac{1 \pm \sqrt{1+8A}}{4} \; , \qquad b = \frac{1 \pm \sqrt{1+8B}}{4} \; ; \qquad (4.5a)$$

two linear system are written as

$$(A - \lambda)A_1 + \lambda A_2 = 0 \; ,$$

$$(A - \lambda')A_3 + \lambda' A_2 = 0 \; ,$$

$$(A - \frac{2+\lambda+\lambda'}{2})A_2 + A_1 + A_3 = 0 \; ; \qquad (4.5b)$$

and

$$(B-\lambda)B_1 - \lambda B_2 = 0 \; ,$$

$$(B-\lambda')B_3 - \lambda' B_2 = 0 \; ,$$

$$(B - \frac{2+\lambda+\lambda'}{2})B_2 - B_1 - B_3 = 0 \; , \qquad (4.5c)$$

Further we get one the same eigenvalue equation for values A and B

$$(A-\lambda)\lambda' + (A-\lambda')\lambda - (A-\lambda)(A-\lambda')(A - \frac{2+\lambda+\lambda'}{2}) = 0 \; ,$$

$$(B-\lambda)\lambda' + (B-\lambda')\lambda - (B-\lambda)(A-\lambda')(B - \frac{2+\lambda+\lambda'}{2}) = 0 \; ; \qquad (4.6)$$

respective solutions are given as

$$A_1 = (A_2)\,\frac{\lambda}{\lambda - A} \; , \qquad A_3 = (A_2)\,\frac{\lambda'}{\lambda' - A} \; ; \qquad (4.7a)$$

$$B_1 = (-B_2)\,\frac{\lambda}{\lambda - B} \; , \qquad B_3 = (-B_2)\,\frac{\lambda'}{\lambda' - B} \; . \qquad (4.7b)$$

Now, let us examine a third order equation (4.6) – for definiteness consider the case of A:

$$2(A-\lambda)\lambda' + 2(A-\lambda')\lambda - (A-\lambda)(A-\lambda')(2A - 2 - \lambda - \lambda') = 0 \; ; \qquad (4.8)$$

the equation arising is symmetric with respect to formal replacement $\lambda \Leftrightarrow \lambda'$. Explicitly the equation read

$$2A(\lambda+\lambda') - 4\lambda\lambda' + [A^2 - A(\lambda+\lambda') + \lambda\lambda'][-2A + 2 + (\lambda+\lambda')] = 0 \qquad \Longrightarrow$$

$$2A(\lambda+\lambda') - 4\lambda\lambda' - 2A^3 + 2A^2 + A^2(\lambda+\lambda')$$

$$+2A^2(\lambda+\lambda') - 2A(\lambda+\lambda') - A(\lambda+\lambda')^2 - 2A\lambda\lambda' + 2\lambda\lambda' + \lambda\lambda'(\lambda+\lambda') = 0 \qquad \Longrightarrow$$

$$-2A^3 + A^2\,[\,2 + 3(\lambda+\lambda')\,] - A\,[\,(\lambda+\lambda')^2 + 2\lambda\lambda'\,] + \lambda\lambda'\,[\,(\lambda+\lambda') - 2\,] = 0 \; . \qquad (4.9a)$$

Remembering on $\lambda' = \lambda - B$, one can introduce other parameters

$$\lambda' - \frac{B}{2} = \lambda + \frac{B}{2} \equiv \Lambda \; ,$$

$$\lambda + \lambda' = 2\Lambda \; , \qquad \lambda\lambda' = \Lambda^2 - \frac{B^2}{4} \; . \tag{4.9b}$$

Then eq. (4.9a) reads

$$A^3 - A^2 \,(3\Lambda + 1) + A\,(3\Lambda^2 - \frac{B^2}{4}) - (\Lambda^2 - \frac{B^2}{4})\,(\Lambda - 1) = 0 \; . \tag{4.9c}$$

It can be presented symbolically as

$$A^3 + aA^2 + bA + c = 0 \; , \tag{4.10a}$$

where

$$a = -(3\Lambda + 1) \; ,$$

$$b = (3\Lambda^2 - \frac{B^2}{4}) \; ,$$

$$c = -(\Lambda^2 - \frac{B^2}{4})\,(\Lambda - 1) \; . \tag{4.10b}$$

Through change in the variable ($A \Longrightarrow Y$)

$$A = Y - \frac{a}{3} = Y + \Lambda + \frac{1}{3} \tag{4.11a}$$

we remove a quadratic term

$$Y^3 + pY + q = 0 \; , \tag{4.11b}$$

where

$$p = -\frac{a^2}{3} + b = -(2\Lambda + \frac{B^2}{4} + \frac{1}{3}) \; ,$$

$$q = \frac{2a^3}{27} - \frac{ab}{3} + c = -(\frac{2}{3}\Lambda + \frac{B^2}{3} + \frac{2}{27}) \; . \tag{4.11c}$$

Note substantial inequalities

$$p < 0, \qquad q < 0, \qquad \mid p \mid > \mid q \mid \; .$$

Formulas, giving solutions of eq. (4.11b) are well known

$$Y = \left[-\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3} \right]^{1/3} + \left[-\frac{q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3} \right]^{1/3} . \qquad (4.12a)$$

Applying (4.12a), one must use correlated roots

$$\alpha = \left[-\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3} \right]^{1/3} \qquad (4.12b)$$

and

$$\beta = \left[-\frac{q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3} \right]^{1/3} \qquad (4.12c)$$

so that the following restriction hold

$$\alpha\beta = -\frac{p}{3} . \qquad (4.12d)$$

Besides, the roots can be searched according to the formulas

$$\begin{aligned} Y_1 &= \alpha_1 + \beta_1 , \\ Y_2 &= -\frac{1}{2}(\alpha_1 + \beta_1) + i\frac{\sqrt{3}}{2}(\alpha_1 - \beta_1) \\ Y_3 &= -\frac{1}{2}(\alpha_1 + \beta_1) - i\frac{\sqrt{3}}{2}(\alpha_1 - \beta_1) \qquad (4.13a) \end{aligned}$$

where α_1 stands for any root in (4.12b), but a root β_1 in (4.12c) must obey

$$\alpha_1\beta_1 = -\frac{p}{3} . \qquad (4.13b)$$

Let us additionally detail expressions (4.13a,b) for three roots. Allowing for

$$\begin{aligned} \alpha &= \left[-q/2 + i\sqrt{(-p/3)^3 - (q/2)^2} \right]^{1/3} \\ &= \left[(-p/3)^{3/2} (\cos\phi + i\sin\phi) \right]^{1/3} \\ &= \sqrt{-p/3} \ \left\{ e^{i\phi/3} , \ e^{i(\phi/3 + 2\pi/3)} , \ e^{i(\phi/3 + 4\pi/3)} \right\} , \qquad (4.14a) \end{aligned}$$

where

$$\cos\phi = \frac{-q/2}{(-p/3)^{3/2}} \;, \qquad \sin\phi = \frac{\sqrt{(-p/3)^3 - (q/2)^2}}{(-p/3)^{3/2}} \;. \tag{4.14b}$$

It is readily to specify the quantity β:

$$\beta = \Big[-q/2 - i\sqrt{(-p/3)^3 - (q/2)^2}\; \Big]^{1/3} =$$

$$= \Big[(-p/3)^{3/2} (\cos\phi - i\sin\phi) \Big]^{1/3} =$$

$$= \sqrt{-p/3}\;\; \Big\{ e^{-i\phi/3} \;,\; e^{i(-\phi/3+2\pi/3)} \;,\; e^{i(-\phi/3+4\pi/3)} \Big\} \;, \tag{4.14a}$$

where

$$\cos\phi = \frac{-q/2}{(-p/3)^{3/2}} \;, \qquad \sin\phi = \frac{\sqrt{(-p/3)^3 - (q/2)^2}}{(-p/3)^{3/2}} \;. \tag{4.14b}$$

As α_1 and β_1 we will take

$$\alpha_1 = \sqrt{-p/3} e^{+i\phi/3} \;, \qquad \beta_1 = \sqrt{-p/3} e^{-i\phi/3} \;;$$

$$\cos\phi = \frac{-q/2}{(-p/3)^{3/2}} \;, \qquad \sin\phi = \frac{\sqrt{(-p/3)^3 - (q/2)^2}}{(-p/3)^{3/2}} \;. \tag{4.15a}$$

And further we readily find

$$\alpha_1 + \beta_1 = 2\sqrt{-p/3}\; \cos\frac{\phi}{3} \;, \qquad \alpha_1 - \beta_1 = 2i\sqrt{-p/3}\; \sin\frac{\phi}{3} \;. \tag{4.15b}$$

Thus, three different (real-valued) roots are determined by the formulas

$$Y_1 = \sqrt{-p/3}\; (2\cos\frac{\phi}{3}) \;,$$

$$Y_2 = \sqrt{-p/3}\; (-\cos\frac{\phi}{3} - \sqrt{3}\sin\frac{\phi}{3}) \;,$$

$$Y_3 = \sqrt{-p/3}\; (-\cos\frac{\phi}{3} + \sqrt{3}\sin\frac{\phi}{3}) \;. \tag{4.16}$$

One can additionally check the results: from the identity

$$Y^3 + pY + q = (Y - Y_1)(Y - Y_2)(Y - Y_3)$$

it follows

$$
\begin{aligned}
0 &= Y_1 + Y_2 + Y_3 \, , \\
p = Y_1 Y_2 + Y_1 Y_3 + Y_2 Y_3 \, , &\qquad q = -Y_1 Y_2 Y_3 \, . \qquad (4.17)
\end{aligned}
$$

First we readily verify two identity

$$
0 = Y_1 + Y_2 + Y_3 \, , \qquad p = Y_1 Y_2 + Y_1 Y_3 + Y_2 Y_3 \, .
$$

Turning to the third ine, leyt us calculate

$$
-Y_1 Y_2 Y_3 = -\frac{2\sqrt{3}}{9} (-p)^{3/2} \left[4 \cos^2 \frac{\phi}{3} - 3 \right] \cos \frac{\phi}{3} \, ; \qquad (4.18a)
$$

further with the help of elementary relation

$$
\cos\alpha \cos\beta = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2} \, ,
$$

we get

$$
\left[4 \cos^2 \frac{\phi}{3} - 3 \right] \cos \frac{\phi}{3} = (-1 + 2 \cos \frac{2\phi}{3}) \cos \frac{\phi}{3} = \cos\phi \, ; \qquad (4.18b)
$$

and thus we prove the third identity (remembering on (4.15a))

$$
-Y_1 Y_2 Y_3 = -\frac{2\sqrt{3}}{9} (-p)^{3/2} \cos\phi = \frac{2\sqrt{3}}{9} (-p)^{3/2} \, \frac{-q/2}{(-p/3)^{3/2}} = -q \, . \quad (4.18c)
$$

Unfortunately we have not gained success in solving the main system of 3 equation in z variable

$$
\begin{aligned}
\left((1 - x^2) \frac{d^2}{dx^2} - x \frac{d}{dx} - \frac{2\lambda}{1 - x^2} \ + 2\epsilon M \right) Z_1 + \frac{2\lambda x}{1 - x^2} \, \bar{Z}_2 = 0 \, , \\
\left((1 - x^2) \frac{d^2}{dx^2} - x \frac{d}{dx} - \frac{2\lambda'}{1 - x^2} \ + 2\epsilon M \right) Z_3 + \frac{2\lambda' x}{1 - x^2} \, \bar{Z}_2 = 0 \, , \\
\left((1 - x^2) \frac{d^2}{dx^2} - x \frac{d}{dx} - \frac{2 + \lambda + \lambda'}{1 - x^2} \ + 2\epsilon M + 1 \right) \bar{Z}_2 + \frac{2x}{1 - x^2} \, (Z_1 + Z_3) = 0 \, .
\end{aligned}
$$

So this analysis canon be considered as completed.

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